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Sets

Set theory is about studying collection of objects. The collection may comprise anything or any abstraction. It can be purely abstract thing like numbers or abstraction of real thing like students studying in class XI in a school. The members of collection can be numbers, letters, titles of books, people, teachers, provinces – virtually anything - even other collections. Further, it need not be finite. For example, a set of integers has infinite members. For a set, only requirement is that the members of a collection are properly defined.

Set

A set is a collection of well defined objects.

In other words, the member of set is clearly identifiable. The terms “object”, “member” or “element” mean same thing and are used interchangeably.

How to represent a set?

A set is denoted by capital letters like “A”, “B”, “C” etc. In choosing a symbol for a set, it is generally convenient to use a capital letter that identifies with the set. For example, it is appropriate to use symbol “V” to represent collection of vowels in English alphabet.

On the other hand, the members or elements of a set are denoted by small letters like “a”,”b”,”c” etc.

Membership of a set is denoted by the symbol “ $\in$” . Its literal meaning is “belongs to”. If an object does not belong to a set, then we convey the same, using symbol “ $\notin$ ”.

$a \in A$: we read this as “a” belongs to set "A".

$a \notin A$: we read this as “a” does not belong to set "A".

The set is represented in two ways :

- Roaster form
- Set builder form

Roaster form

All elements of the set are listed with a comma (“,”) in between and the listing itself is enclosed within braces “{“ and “}”. The order or sequence of elements within the set is not important – though desirable.

The set of numbers, which divide 12, is written as :

$$A = \{1,2,3,4,6,12\}$$

If a pattern or sequence is easily made out, then we can use ellipsis ("...") to represent continuity of such pattern. This type of representation is particularly useful to represent an infinite set. Clearly, sequence of members in this type of representation is important.

The set of even numbers is written as,

$$B = \{2,4,6,8 . . . . . .\}$$

The roaster form is limited in certain circumstance. For example, we can not represent set of real numbers in roaster form. Real numbers is an infinite set, but the elements of this set do not follow a pattern or have a particular sequence. As such, we can not define same with the help of ellipsis.

Every member of the set is unique and distinct. However, we encounter situations in which collection can have repeated elements. For example, the set representing scores of three students can be a set of three numbers one of which is repeated :

$$S = \{80,80,70\}$$

We need to reduce such collection as :

$$\Rightarrow S = \{80,80,70\} = \{80,70\}$$

Set builder form

Collections are often characterized by a common property. We can, therefore, define members of a set in terms of the common property. However, we need to

ensure that objects outside the collection do not have the same common property.

The construction of qualification for the common property is quite flexible. Only thing is that we need to be explicit in what we mean. Generally, we denote the member by a symbol like “x” and then define the membership. Consider the examples :

$$A = \{\text{x: x is a vowel in English alphabet}\}$$

$$B = \{\text{x: x is an integer and } 0 < x < 10\}$$

The roaster equivalents of two sets are :

$$A = \{a, e, i, o, u\}$$

$$B = \{1,2,3,4,5,6,7,8,9\}$$

Can we write set “B” as the one which comprises single digit natural number? Yes. Thus, we can see that there are indeed different ways to define and identify members and hence the flexibility in defining collection.

We should be careful in using words like “and” and “or” in writing qualification for the set. Consider the example here :

$$C = \{\text{x:} x \in Z \text{ and } 2 < x < 4\}$$

Both conditional qualifications are used to determine the collection. The elements of the set as defined above are integers. Thus, the only member of the set is “3”.

Now, let us consider an example, which involves “or” in the qualification,

$$C = \{\text{x: } x \in A \text{ or } x \in B\}$$

The member of this set can be elements belonging to either of two sets "A" and "B". The set consists of elements (i) belonging exclusively to set "A", (ii) elements belonging exclusively to set "B" and (iii) elements common to sets "A" and "B".

Example

Problem 1 : A set in roaster form is given as :

$$A = \left\{ \frac{5^2}{6}, \frac{6^2}{7}, \frac{7^2}{8} \right\}$$

Write the set in “set builder form”.

Solution : We see here that we are dealing with natural numbers. The numerators are square of natural numbers in sequence. The number in denominator is one more than numerator for each member. We can denote natural number by “n”. Clearly, if numerator is “ n^2 ”, then denominator is “n+1”. Therefore, the expression that represent a member of the set is :

$$x = \frac{n^2}{n+1}$$

However, this set is not an infinite set. It has exactly three members. Therefore, we need to specify “n” so that only members of the set are exclusively denoted by the above expression. We see here that “n” is greater than 4, but “n” is less than 8. For representing three elements of the set,

$$5 \leq n \leq 7$$

We can write the set, now, in the builder form as :

$$A = \left\{ x : \quad x = \frac{n^2}{n+1}, \text{where ”n” is a natural number and} \quad 5 \leq n \leq 7 \right\}$$

In set builder form, the sequence within the range is implied. It means that we start with the first valid natural number and proceed sequentially till the last valid natural number.

Some important sets representing numbers

Few key number sets are used regularly in mathematical context. As we use these sets often, it is convenient to have predefined symbols :

- P(prime numbers)

- N (natural numbers)
- Z (integers)
- Q (rational numbers)
- R (real numbers)

We put a superscript “+”, if we want to specify membership of only positive numbers, where appropriate. " Z^+ ", for example, means set of positive integers.

Empty set

An empty set has no member or object. It is denoted by symbol “φ” and is represented by a pair of braces without any member or object.

$$\varphi = \{\}$$

The empty set is also called “null” or “void” set. For example, consider a definition : “the set of integer between 1 and 2”. There is no integer within this range. Hence, the set corresponding to this definition is an empty set. Consider another example :

$$B = \left\{x : \quad x^2 = 4 \quad \text{and x is odd}\right\}$$

An odd integer squared can not be even. Hence, set “B” also does not have any element in it.

There is a bit of paradox here. If the definition does not yield an element, then the set is not well defined. We may be tempted to say that empty set is not a set in the first place. However, there is a practical reason to have an empty set. It enables mathematical operations. We shall find many examples as we study operations on sets.

Equal sets

The members of two equal sets are exactly same. There is nothing more to it. However, we need to know two special aspects of this equality. We mentioned about repetition of elements in a set. We observed that repetition of elements does not change the set. Consider example here :

$$A = \{1,5,5,8,7\} = \{1,5,8,7\}$$

Another point is that sequence does not change the set. Therefore,

$$A = \{1,5,8,7\} = \{5,7,8,1\}$$

In the nutshell, when we have to compare two sets we look for distinct elements only. If they are same, then two sets in question are equal.

Cardinality

Cardinality is the numbers of elements in a set. It is denoted by modulus of set like |A|.

Cardinality

> The cardinality of a set “A” is equal to numbers of elements in the set.

The cardinality of an empty set is zero. The cardinality of a finite set is some positive integers. The cardinality of a number system like integers is infinity. Curiously, the cardinality of some infinite set can be compared. For example, the cardinality of natural numbers is less than that of integers. However, we can not make such deduction for the most case of infinite sets.

Number Systems
Worked examples for converting between binary, decimal and hexadecimal.

The Number Systems

What are the different number systems?
We are all familiar with the decimal system. However, when working with computers, we need to start with the binary system. The reason for this is that computers use gates (or switches) which only have two states, on and off. This is what translates to the 1's and 0's of binary. From there, it is possible to build up to other more useful systems such as the decimal system or the hexadecimal system.

This module contains worked examples of how to convert between the decimal, hexadecimal and binary.

Powers of 10 and 2

Before working through some examples, it will be useful to review how we use the decimal system. The decimal system can express any real rational number using the digits 0-9 and a minus sign. The places of the digits represent the power of ten that is being used. For example:
Equation:

$$321 = 3 \times 10^2 + 2 \times 10^1 + 1 \times 10^0$$

Equation:

$$5023 = 5 \times 10^3 + 0 \times 10^2 + 2 \times 10^1 + 3 \times 10^0$$

In the same way, binary systems use 1's and 0's to express a number:
Equation:

$$\begin{aligned} 23 &= 10111\ (\text{binary}) \\ &= 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 \end{aligned}$$

Binary - Unsigned

The following examples show how to convert between unsigned binary and decimal values.
Exercise:

Problem: What is the decimal value of 10101 ?

Solution:

Know which system you are dealing with. In this case, we are dealing with unsigned binary numbers. Our

range of possible numbers are between 0 and $2^N - 1$.

Write out the sum of each digit multiplied by its correct power of two depending on its position

Equation:

$$\begin{aligned} 10101 &= 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \\ &= 1 \times 2^4 + 1 \times 2^2 + 1 \\ &= 16 + 4 + 1 \\ &= 21 \end{aligned}$$

Exercise:

Problem: Convert 011010 to decimal.

Solution:

Write out the sum of each digit multiplied by its correct power of two:
Equation:

$$\begin{aligned} 011010 &= 0 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 \\ &= 16 + 8 + (2 = 26) \end{aligned}$$

Exercise:

Problem: Convert the decimal number 47 to binary unsigned.

Solution:

Find the largest multiple of two less than or equal to the decimal number

For the decimal number 47, the largest multiple of two is 32 (2^5).

Subtract the largest multiple from the decimal number.

Equation:

$$47 - 32 = 15$$

Repeat Step 1 and 2 until there is no remainder

Equation: $15 - 8 = 7$ **Equation:** $7 - 4 = 3$ **Equation:** $3 - 2 = 1$ **Equation:** $1 - 1 = 0$

Construct the binary number out of the powers of two subtracted from the decimal number

Equation:

$$\begin{aligned} 47 &= 32 + 8 + 4 + 2 + 1 \\ &= 2^5 + 2^3 + 2^2 + 2^1 + 2^0 \\ &= 101111(\text{binary}) \end{aligned}$$

Note: If necessary, you can check your answer by reversing the steps and converting it back to decimal.

Binary Signed

Exercise:

Problem: Convert 11001110 (signed) to decimal value.

Solution:

Check most significant bit to see if it is negative or positive

The most significant bit is 1. This means it is negative.

Sum the powers of two for the bits after the most significant one

Equation:

$$\begin{aligned} 1001110 &= 2^6 + 2^3 + 2^2 + 2^1 \\ &= 64 + 8 + 4 + 2 \\ &= 78 \end{aligned}$$

Attach the sign to the decimal number

Thus the answer is -78.

Exercise:

Problem: Convert -98 to signed binary(8bit).

Solution:

The most significant bit holds the sign value

In this case, the decimal number is negative so the most significant bit is 1.

Subtract the largest multiple of 2 less than or equal to the decimal number

Equation:

$$\begin{aligned} 98 - 2^6 &= 98 - 64 \\ &= 34 \end{aligned}$$

Repeat this step until there is no remainder. Note down the powers of two

Equation:

$$\begin{aligned} 34 - 2^5 &= 34 - 32 \\ &= 2 \end{aligned}$$

Equation:

$$2 - 2^1 = 0$$

Construct the decimal number from the powers of two from the previous steps

Equation:

$$\begin{aligned} -98 &= -\left(2^6 + 2^5 + 2^1\right) \\ &= 11100010 \end{aligned}$$

Exercise:

Problem: Convert 98 to signed binary(8bit).

Solution:

The most significant bit holds the sign value	In this case, the decimal number is negative so the most significant bit is 0.	Since we have already calculated the binary representation for 98, we can use the answer from the previous example. The steps are shown again to illustrate this.
Subtract the largest multiple of 2 less than or equal to the decimal number	**Equation:** $$\begin{aligned} 98 - 2^6 &= 98 - 64 \\ &= 34 \end{aligned}$$	
Repeat this step until there is no remainder. Note down the powers of two	**Equation:** $$\begin{aligned} 34 - 2^5 &= 34 - 32 \\ &= 2 \end{aligned}$$	**Equation:** $$2 - 2^1 = 0$$
Construct the decimal number from the powers of two from the previous steps	**Equation:** $$\begin{aligned} 98 &= 2^6 + 2^5 + 2^1 \\ &= 01100010 \end{aligned}$$	

Binary - Two's Complement

The table below is a refresher for two's complement.

Two's Complement	Decimal
0111	7
0110	6
0101	5

0100	4
0011	3
0010	2
0001	1
0000	0
1111	-1
1110	-2
1101	-3
1100	-4
1011	-5
1010	-6
1001	-7
1000	-8

Two's Complment

Exercise:

Problem: Convert 001011 (two's complement 6-bit) to decimal value.

Solution:

The most significant bit holds the sign value following the same convention as signed binary

In this case, the most significant bit is 0. The number is positive.

Since this number is positive, we can treat the remaining binary digits as per normal

Equation:

$$\begin{aligned} 001011 &= 1011 \\ &= 2^3 + 2^1 + 2^0 \\ &= 8 + 2 + 1 \\ &= 11 \end{aligned}$$

Exercise:

Problem: Convert 111011 (two's complement 6-bit) to decimal value

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Solution:

Check the most significant bit for sign value

The first bit is 1 so the number is negative.

Invert the remaining bits

11011 --> 00100

Add 1 and calculate the decimal value of the number

Equation:

$$\begin{aligned} 00100 + 1 &= 00101 \\ &= 2^2 + 2^0 \\ &= 5 \end{aligned}$$

Add the negative sign to decimal value.

Thus the answer is -5.

Exercise:

Problem: Convert -13 to two's complement 8-bit binary.

Solution:

Use the sign of the decimal value to choose the most significant bit

The number is negative so the most significant bit will be 1.

Convert the decimal value into binary

Equation:

$$\begin{aligned} 13 &= 8 + 4 + 1 \\ &= 2^3 + 2^2 + 2^0 \\ &= 1101 \end{aligned}$$

Subtract 1 and invert the binary numbers

Equation:

$$1101 - 1 = 1100$$

1100 --> 0011

Sign extend the number to the appropriate number of digits

0011 --> 11110011

Hexadecimal

A reference table is attached for conversion between decimal, hexadecimal and binary.

Decimal	Hexadecimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
10	A	1010
11	B	1011
12	C	1100
13	D	1101
14	E	1110
15	F	1111

Hexadecimal Reference

Exercise:

Problem: Convert ABC (hexadecimal) to binary and decimal.

It may sometimes be easier to convert to decimal first and then binary.

Solution:

Convert the letters to their decimal values using the table

ABC --> 10 , 11 , 12

Multiply the decimal values Equation:

by the correct power of 16 based on their significant position. Add the values

$$10 \times 16^2 + 11 \times 16^1 + 12 \times 16^0 = 2560 + 176 + 12 = 2748$$

Convert the decimal values to 4-bit binary numbers

ABC --> 10 + 11 + 12 --> 1010 1011 1100

The binary number can be constructed with these values simply by merging them

1010 1011 1100 --> 101010111100

Exercise:

Problem: Convert 1010011110000001 to its decimal and hexadecimal values.

Solution:

Separate the binary number into blocks of 4 starting from the least significant bit

1010011110000001 --> 1010 0111 1000 0001

Convert each 4-bit block into its hexadecimal value to get the answer

1010 0111 1000 0001 --> A781

It is usually easier to convert from hexadecimal to decimal than from binary to decimal

A781 --> 10, 7, 8, 1

Equation:

$$10 \times 16^3 + 7 \times 16^2 + 8 \times 16^1 + 1 \times 16^0 = 40960 + 1792 + 128 + 1 = 42881$$

Signed Numbers: Absolute Value
This module is from Fundamentals of Mathematics by Denny Burzynski and Wade Ellis, Jr. This module discusses absolute value. By the end of the module students should understand the geometric and algebraic definitions of absolute value.

Section Overview

- Geometric Definition of Absolute Value
- Algebraic Definition of Absolute Value

Geometric Definition of Absolute Value

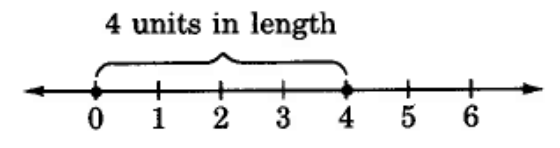
Absolute Value-Geometric Approach
Geometric definition of absolute value:
The **absolute value** of a number a, denoted $|a|$, is the distance from a to 0 on the number line.

Absolute value answers the question of "how far," and not "which way." The phrase "how far" implies "length" and *length is always a nonnegative quantity*. Thus, the absolute value of a number is a nonnegative number.

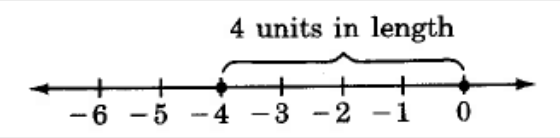
Sample Set A

Determine each value.

Example:
$|4| = 4$



Example:
$|-4|=4$



Example:
$|0|=0$

Example:
$-|5|=-5$. The quantity on the left side of the equal sign is read as "negative the absolute value of 5." The absolute value of 5 is 5. Hence, negative the absolute value of 5 is -5.

Example:
$-|-3|=-3$. The quantity on the left side of the equal sign is read as "negative the absolute value of -3." The absolute value of -3 is 3. Hence, negative the absolute value of -3 is $-(3)=-3$.

Practice Set A

By reasoning geometrically, determine each absolute value.
Exercise:

Problem: $|7|$

Solution:

7

Exercise:

Problem: $|-3|$

Solution:

3

Exercise:

Problem: $|12|$

Solution:

12

Exercise:

Problem: $|0|$

Solution:

0

Exercise:

Problem: $-|9|$

Solution:

-9

Exercise:

Problem: $-|-6|$

Solution:

Algebraic Definition of Absolute Value

From the problems in [link], we can suggest the following algebraic definition of absolute value. Note that the definition has two parts.

Absolute Value—Algebraic Approach
Algebraic definition of absolute value
The absolute value of a number a is

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } < 0 \end{cases}$$

The algebraic definition takes into account the fact that the number a could be either positive or zero $(a \geq 0)$ or negative $(a < 0)$.

1. If the number a is positive or zero $(a \geq 0)$, the upper part of the definition applies. The upper part of the definition tells us that if the number enclosed in the absolute value bars is a nonnegative number, the absolute value of the number is the number itself.
2. The lower part of the definition tells us that if the number enclosed within the absolute value bars is a negative number, the absolute value of the number is the opposite of the number. The opposite of a negative number is a positive number.

Note: The definition says that the vertical absolute value lines may be eliminated only if we know whether the number inside is positive or negative.

Sample Set B

Use the algebraic definition of absolute value to find the following values.

Example:
$|\ 8\ |$. The number enclosed within the absolute value bars is a nonnegative number, so the upper part of the definition applies. This part says that the absolute value of 8 is 8 itself.
$|\ 8\ | = 8$

Example:
$|\ -3\ |$. The number enclosed within absolute value bars is a negative number, so the lower part of the definition applies. This part says that the absolute value of -3 is the opposite of -3, which is $-(-3)$. By the definition of absolute value and the double-negative property,
$|\ -3\ | = -(-3) = 3$

Practice Set B

Use the algebraic definition of absolute value to find the following values.
Exercise:

Problem: $|\ 7\ |$

Solution:

7

Exercise:

Problem: $|\ 9\ |$

Solution:

9

Exercise:

Problem: $|-12|$

Solution:

12

Exercise:

Problem: $|-5|$

Solution:

5

Exercise:

Problem: $-|8|$

Solution:

-8

Exercise:

Problem: $-|1|$

Solution:

-1

Exercise:

Problem: $-|-52|$

Solution:

-52

Exercise:

Problem: $-\left|-31\right|$

Solution:

-31

Exercises

Determine each of the values.

Exercise:

Problem: $\left|5\right|$

Solution:

5

Exercise:

Problem: $\left|3\right|$

Exercise:

Problem: $\left|6\right|$

Solution:

6

Exercise:

Problem: $\left|-9\right|$

Exercise:

Problem: $|-1|$

Solution:

1

Exercise:

Problem: $|-4|$

Exercise:

Problem: $-|3|$

Solution:

-3

Exercise:

Problem: $-|7|$

Exercise:

Problem: $-|-14|$

Solution:

-14

Exercise:

Problem: $|0|$

Exercise:

Problem: $|-26|$

Solution:

26

Exercise:

Problem: $-\,|\,-26\,|$

Exercise:

Problem: $-(-\,|\,4\,|)$

Solution:

4

Exercise:

Problem: $-(-\,|\,2\,|)$

Exercise:

Problem: $-(-\,|\,-6\,|)$

Solution:

6

Exercise:

Problem: $-(-\,|\,-42\,|)$

Exercise:

Problem: $|\,5\,|-|\,-2\,|$

Solution:

3

Exercise:

Problem: $|\ -2\ |^3$

Exercise:

Problem: $|\ -(2 \cdot 3)\ |$

Solution:

6

Exercise:

Problem: $|\ -2\ | - |\ -9\ |$

Exercise:

Problem: $(|\ -6\ | + |\ 4\ |)^2$

Solution:

100

Exercise:

Problem: $(|\ -1\ | - |\ 1\ |)^3$

Exercise:

Problem: $(|\ 4\ | + |\ -6\ |)^2 - (|\ -2\ |)^3$

Solution:

92

Exercise:

Problem: $-[|-10|-6]^2$

Exercise:

Problem: $-\left\{-[-\mid -4 \mid+\mid -3 \mid]^3\right\}^2$

Solution:

-1

Exercise:

Problem:

A Mission Control Officer at Cape Canaveral makes the statement “lift-off, *T* minus 50 seconds.” How long is it before lift-off?

Exercise:

Problem:

Due to a slowdown in the industry, a Silicon Valley computer company finds itself in debt $2,400,000. Use absolute value notation to describe this company’s debt.

Solution:

$-\$ \mid -2{,}400{,}000 \mid$

Exercise:

Problem:

A particular machine is set correctly if upon action its meter reads 0. One particular machine has a meter reading of -1.6 upon action. How far is this machine off its correct setting?

Exercises for Review

Exercise:

Problem: ([link]) Find the sum: $\frac{9}{70} + \frac{5}{21} + \frac{8}{15}$.

Solution:

$\frac{9}{10}$

Exercise:

Problem: ([link]) Find the value of $\frac{\frac{3}{10}+\frac{4}{12}}{\frac{19}{20}}$.

Exercise:

Problem: ([link]) Convert $3.2\frac{3}{5}$ to a fraction.

Solution:

$3\frac{13}{50}$ or $\frac{163}{50}$

Exercise:

Problem:

([link]) The ratio of acid to water in a solution is $\frac{3}{8}$. How many mL of acid are there in a solution that contain 112 mL of water?

Exercise:

Problem: ([link]) Find the value of $-6-(-8)$.

Solution:

2

Arithmetic Review: Factors, Products, and Exponents
This module is from Elementary Algebra by Denny Burzynski and Wade Ellis, Jr. This chapter contains many examples of arithmetic techniques that are used directly or indirectly in algebra. Since the chapter is intended as a review, the problem-solving techniques are presented without being developed. Therefore, no work space is provided, nor does the chapter contain all of the pedagogical features of the text. As a review, this chapter can be assigned at the discretion of the instructor and can also be a valuable reference tool for the student.

Overview

- Factors
- Exponential Notation

Factors

Let's begin our review of arithmetic by recalling the meaning of multiplication for whole numbers (the counting numbers and zero).

Multiplication
Multiplication is a description of repeated addition.

In the addition

$$7 + 7 + 7 + 7$$

the number 7 is repeated as an **addend*** 4 **times.** Therefore, we say we have **four times seven** and describe it by writing

$$4 \cdot 7$$

The raised dot between the numbers 4 and 7 indicates multiplication. The dot directs us to multiply the two numbers that it separates. In algebra, the dot is preferred over the symbol $\times$ to denote multiplication because the letter x is often used to represent a number. Thus,

$$4 \cdot 7 = 7 + 7 + 7 + 7$$

Factors and Products
In a multiplication, the numbers being multiplied are called **factors.** The result of a multiplication is called the **product.** For example, in the multiplication

$$4 \cdot 7 = 28$$

the numbers 4 and 7 are factors, and the number 28 is the product. We say that 4 and 7 are factors of 28. (They are not the only factors of 28. Can you think of others?)

Now we know that

$$(\text{factor}) \cdot (\text{factor}) \quad = \quad \text{product}$$

This indicates that a first number is a factor of a second number if the first number divides into the second number with no remainder. For example, since

$$4 \cdot 7 = 28$$

both 4 and 7 are factors of 28 since both 4 and 7 divide into 28 with no remainder.

Exponential Notation

Quite often, a particular number will be repeated as a factor in a multiplication. For example, in the multiplication

$$7 \cdot 7 \cdot 7 \cdot 7$$

the number 7 is repeated as a factor 4 times. We describe this by writing 7^4. Thus,

$7 \cdot 7 \cdot 7 \cdot 7 = 7^4$

The repeated factor is the lower number (the base), and the number recording how many times the factor is repeated is the higher number (the superscript). The superscript number is called an **exponent.**

Exponent
An **exponent** is a number that records how many times the number to which it is attached occurs as a factor in a multiplication.

Sample Set A

For Examples 1, 2, and 3, express each product using exponents.

Example:
$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$. Since 3 occurs as a factor 6 times,

$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^6$

Example:
$8 \cdot 8$. Since 8 occurs as a factor 2 times,

$8 \cdot 8 = 8^2$

Example:
$5 \cdot 5 \cdot 5 \cdot 9 \cdot 9$. Since 5 occurs as a factor 3 times, we have 5^3. Since 9 occurs as a factor 2 times, we have 9^2. We should see the following replacements.

$5 \cdot 5 \cdot 5 \cdot 9 \cdot 9$
$\quad 5^3 \qquad\quad 9^2$

Then we have

$5 \cdot 5 \cdot 5 \cdot 9 \cdot 9 = 5^3 \cdot 9^2$

Example:
Expand 3^5. The base is 3 so it is the repeated factor. The exponent is 5 and it records the number of times the base 3 is repeated. Thus, 3 is to be repeated as a factor 5 times.

$3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

Example:
Expand $6^2 \cdot 10^4$. The notation $6^2 \cdot 10^4$ records the following two facts: 6 is to be repeated as a factor 2 times and 10 is to be repeated as a factor 4 times. Thus,

$6^2 \cdot 10^4 = 6 \cdot 6 \cdot 10 \cdot 10 \cdot 10 \cdot 10$

Exercises

For the following problems, express each product using exponents.
Exercise:

Problem: $8 \cdot 8 \cdot 8$

Solution:

8^3

Exercise:

Problem: $12 \cdot 12 \cdot 12 \cdot 12 \cdot 12$

Exercise:

Problem: $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$

Solution:

5^{7}

Exercise:

Problem: $1 \cdot 1$

Exercise:

Problem: $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 4 \cdot 4$

Solution:

$3^{5} \cdot 4^{2}$

Exercise:

Problem: $8 \cdot 8 \cdot 8 \cdot 15 \cdot 15 \cdot 15 \cdot 15$

Exercise:

Problem: $2 \cdot 2 \cdot 2 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9$

Solution:

$2^{3} \cdot 9^{8}$

Exercise:

Problem: $3 \cdot 3 \cdot 10 \cdot 10 \cdot 10$

Exercise:

Problem:

Suppose that the letters x and y are each used to represent numbers. Use exponents to express the following product.

$x \cdot x \cdot x \cdot y \cdot y$

Solution:

$x^3 \cdot y^2$

Exercise:

Problem:

Suppose that the letters x and y are each used to represent numbers. Use exponents to express the following product.

$x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y$

For the following problems, expand each product (do not compute the actual value).

Exercise:

Problem: 3^4

Solution:

$3 \cdot 3 \cdot 3 \cdot 3$

Exercise:

Problem: 4^3

Exercise:

Problem: 2^5

Solution:

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

Exercise:

Problem: 9^6

Exercise:

Problem: $5^3 \cdot 6^2$

Solution:

$5 \cdot 5 \cdot 5 \cdot 6 \cdot 6$

Exercise:

Problem: $2^7 \cdot 3^4$

Exercise:

Problem: $x^4 \cdot y^4$

Solution:

$x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y$

Exercise:

Problem: $x^6 \cdot y^2$

For the following problems, specify all the whole number factors of each number. For example, the complete set of whole number factors of 6 is 1, 2, 3, 6.

Exercise:

Problem: 20

Solution:

$1,\ 2,\ 4,\ 5,\ 10,\ 20$

Exercise:

Problem: 14

Exercise:

Problem: 12

Solution:

$1,\ 2,\ 3,\ 4,\ 6,\ 12$

Exercise:

Problem: 30

Exercise:

Problem: 21

Solution:

$1,\ 3,\ 7,\ 21$

Exercise:

Problem: 45

Exercise:

Problem: 11

Solution:

1, 11

Exercise:

Problem:17

Exercise:

Problem:19

Solution:

1, 19

Exercise:

Problem:2

Arithmetic Review: Prime Factorization

This module is from Elementary Algebra by Denny Burzynski and Wade Ellis, Jr. This chapter contains many examples of arithmetic techniques that are used directly or indirectly in algebra. Since the chapter is intended as a review, the problem-solving techniques are presented without being developed. Therefore, no work space is provided, nor does the chapter contain all of the pedagogical features of the text. As a review, this chapter can be assigned at the discretion of the instructor and can also be a valuable reference tool for the student.

Overview

- Prime And Composite Numbers
- The Fundamental Principle Of Arithmetic
- The Prime Factorization Of A Whole Number

Prime And Composite Numbers

Notice that the only factors of 7 are 1 and 7 itself, and that the only factors of 23 are 1 and 23 itself.

Prime Number

A whole number greater than 1 whose only whole number factors are itself and 1 is called a **prime number.**

The first seven prime numbers are

2, 3, 5, 7, 11, 13, and 17

The number 1 is not considered to be a prime number, and the number 2 is the first and only even prime number.

Many numbers have factors other than themselves and 1. For example, the factors of 28 are 1, 2, 4, 7, 14, and 28 (since each of these whole numbers and only these whole numbers divide into 28 without a remainder).

Composite Numbers

A whole number that is composed of factors other than itself and 1 is called a **composite number.** Composite numbers are not prime numbers.

Some composite numbers are 4, 6, 8, 10, 12, and 15.

The Fundamental Principle Of Arithmetic

Prime numbers are very important in the study of mathematics. We will use them soon in our study of fractions. We will now, however, be introduced to an important mathematical principle.

The Fundamental Principle of Arithmetic
Except for the order of the factors, every whole number, other than 1, can be factored in one and only one way as a product of prime numbers.

Prime Factorization
When a number is factored so that all its factors are prime numbers, the factorization is called the **prime factorization** of the number.

Sample Set A

Example:
Find the prime factorization of 10.

$10 = 2 \cdot 5$

Both 2 and 5 are prime numbers. Thus, $2 \cdot 5$ is the prime factorization of 10.

Example:
Find the prime factorization of 60.

$$
\begin{aligned}
60 &= 2 \cdot 30 && 30 \text{ is not prime. } 30 = 2 \cdot 15 \\
&= 2 \cdot 2 \cdot 15 && 15 \text{ is not prime. } 15 = 3 \cdot 5 \\
&= 2 \cdot 2 \cdot 3 \cdot 5 && \text{We'll use exponents. } 2 \cdot 2 = 2^2 \\
&= 2^2 \cdot 3 \cdot 5
\end{aligned}
$$

The numbers 2, 3, and 5 are all primes. Thus, $2^2 \cdot 3 \cdot 5$ is the prime factorization of 60.

Example:
Find the prime factorization of 11.

11 is a prime number. Prime factorization applies only to composite numbers.

The Prime Factorization Of A Whole Number

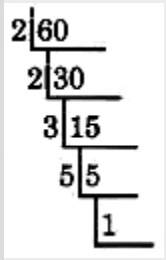
The following method provides a way of finding the prime factorization of a whole number. The examples that follow will use the method and make it more clear.

1. Divide the number repeatedly by the smallest prime number that will divide into the number without a remainder.
2. When the prime number used in step 1 no longer divides into the given number without a remainder, repeat the process with the next largest prime number.
3. Continue this process until the quotient is 1.
4. The prime factorization of the given number is the product of all these prime divisors.

Sample Set B

Example:
Find the prime factorization of 60.

Since 60 is an even number, it is divisible by 2. We will repeatedly divide by 2 until we no longer can (when we start getting a remainder). We shall divide in the following way.



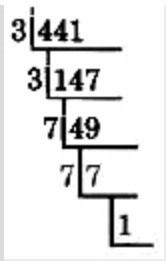
30 is divisible by 2 again.
15 is not divisible by 2, but is divisible by 3, the next largest prime.
5 is not divisible by 3, but is divisible by 5, the next largest prime.
The quotient is 1 so we stop the division process.

The prime factorization of 60 is the product of all these divisors.

$60 = 2 \cdot 2 \cdot 3 \cdot 5$ We will use exponents when possible.
$60 = 2^2 \cdot 3 \cdot 5$

Example:
Find the prime factorization of 441.

Since 441 is an odd number, it is not divisible by 2. We'll try 3, the next largest prime.



147 is divisible by 3.
49 is not divisible by 3 nor by 5, but by 7.
7 is divisible by 7.
The quotient is 1 so we stop the division process.

The prime factorization of 441 is the product of all the divisors.

$441 = 3 \cdot 3 \cdot 7 \cdot 7$ We will use exponents when possible.
$441 = 3^2 \cdot 7^2$

Exercises

For the following problems, determine which whole numbers are prime and which are composite.

Exercise:

Problem: 23

Solution:

prime

Exercise:

Problem: 25

Exercise:

Problem:27

Solution:

composite

Exercise:

Problem:2

Exercise:

Problem:3

Solution:

prime

Exercise:

Problem:5

Exercise:

Problem:7

Solution:

prime

Exercise:

Problem:9

Exercise:

Problem:11

Solution:

prime

Exercise:

Problem:34

Exercise:

Problem:55

Solution:

composite

Exercise:

Problem:63

Exercise:

Problem:1044

Solution:

composite

Exercise:

Problem:339

Exercise:

Problem:209

Solution:

composite

For the following problems, find the prime factorization of each whole number. Use exponents on repeated factors.

Exercise:

Problem: 26

Exercise:

Problem: 38

Solution:

$2 \cdot 19$

Exercise:

Problem: 54

Exercise:

Problem: 62

Solution:

$2 \cdot 31$

Exercise:

Problem: 56

Exercise:

Problem: 176

Solution:

$2^4 \cdot 11$

Exercise:

Problem: 480

Exercise:

Problem: 819

Solution:

$3^2 \cdot 7 \cdot 13$

Exercise:

Problem: 2025

Exercise:

Problem: 148,225

Solution:

$5^2 \cdot 7^2 \cdot 11^2$

Arithmetic Review: The Least Common Multiple

This module is from Elementary Algebra by Denny Burzynski and Wade Ellis, Jr. This chapter contains many examples of arithmetic techniques that are used directly or indirectly in algebra. Since the chapter is intended as a review, the problem-solving techniques are presented without being developed. Therefore, no work space is provided, nor does the chapter contain all of the pedagogical features of the text. As a review, this chapter can be assigned at the discretion of the instructor and can also be a valuable reference tool for the student.

Overview

- Multiples
- Common Multiples
- The Least Common Multiple (LCM)
- Finding The Least Common Multiple

Multiples

Multiples

When a whole number is multiplied by other whole numbers, with the exception of Multiples zero, the resulting products are called **multiples** of the given whole number.

Multiples of 2		**Multiples of 3**		**Multiples of 8**		**Multiples of 10**
$2 \cdot 1 = 2$		$3 \cdot 1 = 3$		$8 \cdot 1 = 8$		$10 \cdot 1 = 10$
$2 \cdot 2 = 4$		$3 \cdot 2 = 6$		$8 \cdot 2 = 16$		$10 \cdot 2 = 20$
$2 \cdot 3 = 6$		$3 \cdot 3 = 9$		$8 \cdot 3 = 24$		$10 \cdot 3 = 30$

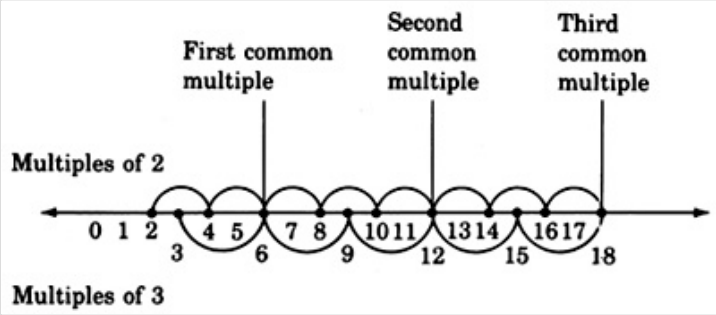
$2 \cdot 4 = 8$		$3 \cdot 4 = 12$		$8 \cdot 4 = 32$		$10 \cdot 4 = 40$
$2 \cdot 5 = 10$		$3 \cdot 5 = 15$		$8 \cdot 5 = 40$		$10 \cdot 5 = 50$
$\cdots$		$\cdots$		$\cdots$		$\cdots$

Common Multiples

There will be times when we are given two or more whole numbers and we will need to know if there are any multiples that are common to each of them. If there are, we will need to know what they are. For example, some of the multiples that are common to 2 and 3 are 6, 12, and 18.

Sample Set A

Example:
We can visualize common multiples using the number line.


First common multiple
Second common multiple
Third common multiple
Multiples of 2
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
Multiples of 3

Notice that the common multiples can be divided by both whole numbers.

The Least Common Multiple (LCM)

Notice that in our number line visualization of common multiples (above) the first common multiple is also the smallest, or **least common multiple,** abbreviated by **LCM.**

Least Common Multiple
The **least common multiple, LCM,** of two or more whole numbers is the smallest whole number that each of the given numbers will divide into without a remainder.

Finding The Least Common Multiple

Finding the LCM
To find the LCM of two or more numbers,

1. Write the prime factorization of each number, using exponents on repeated factors.
2. Write each base that appears in each of the prime factorizations.
3. To each base, attach the largest exponent that appears on it in the prime factorizations.
4. The LCM is the product of the numbers found in step 3.

Sample Set B

Find the LCM of the following number.

Example:
9 and 12

1. $9 = 3 \cdot 3 = 3^2$
   $12 = 2 \cdot 6 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3$
2. The bases that appear in the prime factorizations are 2 and 3.
3. The largest exponents appearing on 2 and 3 in the prime factorizations are, respectively, 2 and 2 (or 2^2 from 12, and 3^2 from 9).
4. The LCM is the product of these numbers.

   $\text{LCM} = 2^2 \cdot 3^2 = 4 \cdot 9 = 36$

Thus, 36 is the smallest number that both 9 and 12 divide into without remainders.

Example:
90 and 630

1. $$\begin{aligned} 90 &= 2 \cdot 45 = 2 \cdot 3 \cdot 15 = 2 \cdot 3 \cdot 3 \cdot 5 = 2 \cdot 3^2 \cdot 5 \\ 630 &= 2 \cdot 315 = 2 \cdot 3 \cdot 105 = 2 \cdot 3 \cdot 3 \cdot 35 = 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 \\ &= 2 \cdot 3^2 \cdot 5 \cdot 7 \end{aligned}$$
2. The bases that appear in the prime factorizations are 2, 3, 5, and 7.
3. The largest exponents that appear on 2, 3, 5, and 7 are, respectively, 1, 2, 1, and 1.

   2^1 from either 90 or 630

   3^2 from either 90 or 630

   5^1 from either 90 or 630

   7^1 from 630
4. The LCM is the product of these numbers.

   $$\text{LCM} = 2 \cdot 3^2 \cdot 5 \cdot 7 = 2 \cdot 9 \cdot 5 \cdot 7 = 630$$

Thus, 630 is the smallest number that both 90 and 630 divide into with no remainders.

Example:
33, 110, and 484

1. $$\begin{aligned} 33 &= 3 \cdot 11 \\ 110 &= 2 \cdot 55 = 2 \cdot 5 \cdot 11 \\ 484 &= 2 \cdot 242 = 2 \cdot 2 \cdot 121 = 2 \cdot 2 \cdot 11 \cdot 11 = 2^2 \cdot 11^2 \end{aligned}$$
2. The bases that appear in the prime factorizations are 2, 3, 5, and 11.
3. The largest exponents that appear on 2, 3, 5, and 11 are, respectively, 2, 1, 1, and 2.

2^2 from 484
3^1 from 33
5^1 from 110
11^2 from 484

4. The LCM is the product of these numbers.

$$\begin{aligned} \text{LCM} &= 2^2 \cdot 3 \cdot 5 \cdot 11^2 \\ &= 4 \cdot 3 \cdot 5 \cdot 121 \\ &= 7260 \end{aligned}$$

Thus, 7260 is the smallest number that 33, 110, and 484 divide into without remainders.

Exercises

For the following problems, find the least common multiple of given numbers.

Exercise:

Problem: 8, 12

Solution:

$2^3 \cdot 3$

Exercise:

Problem: 8, 10

Exercise:

Problem: 6, 12

Solution:

$2^2 \cdot 3$

Exercise:

Problem:9, 18

Exercise:

Problem:5, 6

Solution:

$2 \cdot 3 \cdot 5$

Exercise:

Problem:7, 9

Exercise:

Problem:28, 36

Solution:

$2^2 \cdot 3^2 \cdot 7$

Exercise:

Problem:24, 36

Exercise:

Problem:28, 42

Solution:

$2^2 \cdot 3 \cdot 7$

Exercise:

Problem:20, 24

Exercise:

Problem:25, 30

Solution:

$2 \cdot 3 \cdot 5^2$

Exercise:

Problem:24, 54

Exercise:

Problem:16, 24

Solution:

$2^4 \cdot 3$

Exercise:

Problem:36, 48

Exercise:

Problem:15, 21

Solution:

$3 \cdot 5 \cdot 7$

Exercise:

Problem:7, 11, 33

Exercise:

Problem:8, 10, 15

Solution:

$2^3 \cdot 3 \cdot 5$

Exercise:

Problem:4, 5, 21

Exercise:

Problem: 45, 63, 98

Solution:

$2 \cdot 3^2 \cdot 5 \cdot 7^2$

Exercise:

Problem: 15, 25, 40

Exercise:

Problem: 12, 16, 20

Solution:

$2^4 \cdot 3 \cdot 5$

Exercise:

Problem: 12, 16, 24

Exercise:

Problem: 12, 16, 24, 36

Solution:

$2^4 \cdot 3^2$

Exercise:

Problem: 6, 9, 12, 18

Exercise:

Problem: 8, 14, 28, 32

Solution:

$2^5 \cdot 7$

Arithmetic Review: Equivalent Fractions

This module is from Elementary Algebra by Denny Burzynski and Wade Ellis, Jr. This chapter contains many examples of arithmetic techniques that are used directly or indirectly in algebra. Since the chapter is intended as a review, the problem-solving techniques are presented without being developed. Therefore, no work space is provided, nor does the chapter contain all of the pedagogical features of the text. As a review, this chapter can be assigned at the discretion of the instructor and can also be a valuable reference tool for the student.

Overview

- Equivalent Fractions
- Reducing Fractions To Lowest Terms
- Raising Fractions To Higher Terms

Equivalent Fractions

Equivalent Fractions

Fractions that have the same value are called **equivalent fractions.**

For example, $\frac{2}{3}$ and $\frac{4}{6}$ represent the same part of a whole quantity and are therefore equivalent. Several more collections of equivalent fractions are listed below.

Example:
$\frac{15}{25}, \frac{12}{20}, \frac{3}{5}$

Example:
$\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}$

Example:
$\frac{7}{6}, \frac{14}{12}, \frac{21}{18}, \frac{28}{24}, \frac{35}{30}$

Reducing Fractions To Lowest Terms

Reduced to Lowest Terms

It is often useful to convert one fraction to an equivalent fraction that has reduced values in the numerator and denominator. When a fraction is converted to an equivalent fraction that has the smallest numerator and denominator in the collection of equivalent fractions, it is said to be **reduced to lowest terms.** The conversion process is called **reducing a fraction.**

We can reduce a fraction to lowest terms by

1. Expressing the numerator and denominator as a product of prime numbers. (Find the prime factorization of the numerator and denominator. See Section ([link]) for this technique.)
2. Divide the numerator and denominator by all common factors. (This technique is commonly called “cancelling.”)

Sample Set A

Reduce each fraction to lowest terms.

Example:

$$\begin{aligned}\frac{6}{18} &= \frac{2\cdot 3}{2\cdot 3\cdot 3}\\ &= \frac{\cancel{2}\cdot\cancel{3}}{\cancel{2}\cdot\cancel{3}\cdot 3} \qquad 2 \text{ and } 3 \text{ are common factors.}\\ &= \frac{1}{3}\end{aligned}$$

Example:

$$\begin{aligned}\frac{16}{20} &= \frac{2\cdot 2\cdot 2\cdot 2}{2\cdot 2\cdot 5}\\ &= \frac{\cancel{2}\cdot\cancel{2}\cdot 2\cdot 2}{\cancel{2}\cdot\cancel{2}\cdot 5} \qquad 2 \text{ is the only common factor.}\\ &= \frac{4}{5}\end{aligned}$$

Example:

$$\begin{aligned}\frac{56}{70} &= \frac{2\cdot 4\cdot 7}{2\cdot 5\cdot 7}\\ &= \frac{\cancel{2}\cdot 4\cdot\cancel{7}}{\cancel{2}\cdot 5\cdot\cancel{7}} \qquad 2 \text{ and } 7 \text{ are common factors.}\\ &= \frac{4}{5}\end{aligned}$$

Example:

$$\frac{8}{15} = \frac{2\cdot 2\cdot 2}{3\cdot 5} \qquad \text{There are no common factors.}$$

Thus, $\frac{8}{15}$ is reduced to lowest terms.

Raising a Fraction to Higher Terms

Equally important as reducing fractions is **raising fractions to higher terms.** Raising a fraction to higher terms is the process of constructing an equivalent fraction that has higher values in the numerator and denominator. The higher, equivalent fraction is constructed by multiplying the original fraction by 1.

Notice that $\frac{3}{5}$ and $\frac{9}{15}$ are equivalent, that is $\frac{3}{5} = \frac{9}{15}$. Also,

$$\frac{3}{5} \cdot 1 = \frac{3}{5} \cdot \frac{3}{3} = \frac{3 \cdot 3}{5 \cdot 3} = \frac{9}{15}$$

$$1 = \frac{3}{3}$$

This observation helps us suggest the following method for raising a fraction to higher terms.

Raising a Fraction to Higher Terms
A fraction can be raised to higher terms by multiplying both the numerator and denominator by the same nonzero number.

For example, $\frac{3}{4}$ can be raised to $\frac{24}{32}$ by multiplying both the numerator and denominator by 8, that is, multiplying by 1 in the form $\frac{8}{8}$.

$$\frac{3}{4} = \frac{3 \cdot 8}{4 \cdot 8} = \frac{24}{32}$$

How did we know to choose 8 as the proper factor? Since we wish to convert 4 to 32 by multiplying it by some number, we know that 4 must be a factor of 32. This means that 4 divides into 32. In fact, $32 \div 4 = 8$. We divided the original denominator into the new, specified denominator to obtain the proper factor for the multiplication.

Sample Set B

Determine the missing numerator or denominator.

Example:
$\frac{3}{7} = \frac{?}{35}$. Divide the original denominator, 7, into the new denominator, 35. $35 \div 7 = 5$. Multiply the original numerator by 5.
$\frac{3}{7} = \frac{3 \cdot 5}{7 \cdot 5} = \frac{15}{35}$

Example:
$\frac{5}{6} = \frac{45}{?}$. Divide the original numerator, 5, into the new numerator, 45. $45 \div 5 = 9$. Multiply the original denominator by 9.
$\frac{5}{6} = \frac{5 \cdot 9}{6 \cdot 9} = \frac{45}{54}$

Exercises

For the following problems, reduce, if possible, each fraction lowest terms.
Exercise:

Problem: $\frac{6}{8}$

Solution:

$\frac{3}{4}$

Exercise:

Problem: $\frac{5}{10}$

Exercise:

Problem: $\frac{6}{14}$

Solution:

$\frac{3}{7}$

Exercise:

Problem: $\frac{4}{14}$

Exercise:

Problem: $\frac{18}{12}$

Solution:

$\frac{3}{2}$

Exercise:

Problem: $\frac{20}{8}$

Exercise:

Problem: $\frac{10}{6}$

Solution:

$\frac{5}{3}$

Exercise:

Problem: $\frac{14}{4}$

Exercise:

Problem: $\frac{10}{12}$

Solution:

$\frac{5}{6}$

Exercise:

Problem: $\frac{32}{28}$

Exercise:

Problem: $\frac{36}{10}$

Solution:

$\frac{18}{5}$

Exercise:

Problem: $\frac{26}{60}$

Exercise:

Problem: $\frac{12}{18}$

Solution:

$\frac{2}{3}$

Exercise:

Problem: $\frac{18}{27}$

Exercise:

Problem: $\frac{18}{24}$

Solution:

$\frac{3}{4}$

Exercise:

Problem: $\frac{32}{40}$

Exercise:

Problem: $\frac{11}{22}$

Solution:

$\frac{1}{2}$

Exercise:

Problem: $\frac{17}{51}$

Exercise:

Problem: $\frac{27}{81}$

Solution:

$\frac{1}{3}$

Exercise:

Problem: $\frac{16}{42}$

Exercise:

Problem: $\frac{39}{13}$

Solution:

3

Exercise:

Problem: $\frac{44}{11}$

Exercise:

Problem: $\frac{121}{132}$

Solution:

$\frac{11}{12}$

Exercise:

Problem: $\frac{30}{105}$

Exercise:

Problem: $\frac{108}{76}$

Solution:

$\frac{27}{19}$

For the following problems, determine the missing numerator or denominator.

Exercise:

Problem: $\frac{1}{3} = \frac{?}{12}$

Exercise:

Problem: $\frac{1}{5} = \frac{?}{30}$

Solution:

6

Exercise:

Problem: $\frac{3}{3} = \frac{?}{9}$

Exercise:

Problem: $\frac{3}{4} = \frac{?}{16}$

Solution:

12

Exercise:

Problem: $\frac{5}{6} = \frac{?}{18}$

Exercise:

Problem: $\frac{4}{5} = \frac{?}{25}$

Solution:

20

Exercise:

Problem: $\frac{1}{2} = \frac{4}{?}$

Exercise:

Problem: $\frac{9}{25} = \frac{27}{?}$

Solution:

75

Exercise:

Problem: $\frac{3}{2} = \frac{18}{?}$

Exercise:

Problem: $\frac{5}{3} = \frac{80}{?}$

Solution:

48

Arithmetic Review: Operations with Fractions

This module is from Elementary Algebra by Denny Burzynski and Wade Ellis, Jr. This chapter contains many examples of arithmetic techniques that are used directly or indirectly in algebra. Since the chapter is intended as a review, the problem-solving techniques are presented without being developed. Therefore, no work space is provided, nor does the chapter contain all of the pedagogical features of the text. As a review, this chapter can be assigned at the discretion of the instructor and can also be a valuable reference tool for the student.

Overview

- Multiplication of Fractions
- Division of Fractions
- Addition and Subtraction of Fractions

Multiplication of Fractions

Multiplication of Fractions

To multiply two fractions, multiply the numerators together and multiply the denominators together. Reduce to lowest terms if possible.

Example:

For example, multiply $\frac{3}{4} \cdot \frac{1}{6}$.

$$\begin{aligned} \frac{3}{4} \cdot \frac{1}{6} &= \frac{3 \cdot 1}{4 \cdot 6} \\ &= \frac{3}{24} && \text{Now reduce.} \\ &= \frac{3 \cdot 1}{2 \cdot 2 \cdot 2 \cdot 3} \\ &= \frac{\cancel{3} \cdot 1}{2 \cdot 2 \cdot 2 \cdot \cancel{3}} && 3 \text{ is the only common factor.} \\ &= \frac{1}{8} \end{aligned}$$

Notice that we since had to reduce, we nearly started over again with the original two fractions. If we factor first, then cancel, then multiply, we will save time and energy and still obtain the correct product.

Sample Set A

Perform the following multiplications.

Example:

$$\begin{aligned}
\frac{1}{4}\cdot\frac{8}{9} &= \frac{1}{2\cdot 2}\cdot\frac{2\cdot 2\cdot 2}{3\cdot 3} \\
&= \frac{1}{\cancel{2}\cdot\cancel{2}}\cdot\frac{\cancel{2}\cdot\cancel{2}\cdot 2}{3\cdot 3} \qquad 2 \text{ is a common factor.} \\
&= \frac{1}{1}\cdot\frac{2}{3\cdot 3} \\
&= \frac{1\cdot 2}{1\cdot 3\cdot 3} \\
&= \frac{2}{9}
\end{aligned}$$

Example:

$$\begin{aligned}
\frac{3}{4}\cdot\frac{8}{9}\cdot\frac{5}{12} &= \frac{3}{2\cdot 2}\cdot\frac{2\cdot 2\cdot 2}{3\cdot 3}\cdot\frac{5}{2\cdot 2\cdot 3} \\
&= \frac{\cancel{3}}{\cancel{2}\cdot\cancel{2}}\cdot\frac{\cancel{2}\cdot\cancel{2}\cdot\cancel{2}}{\cancel{3}\cdot 3}\cdot\frac{5}{\cancel{2}\cdot 2\cdot 3} \qquad 2 \text{ and } 3 \text{ are common factors.} \\
&= \frac{1\cdot 1\cdot 5}{3\cdot 2\cdot 3} \\
&= \frac{5}{18}
\end{aligned}$$

Division of Fractions

Reciprocals
Two numbers whose product is 1 are **reciprocals** of each other. For example, since $\frac{4}{5}\cdot\frac{5}{4}=1$, $\frac{4}{5}$ and $\frac{5}{4}$ are reciprocals of each other. Some other pairs of reciprocals are listed below.

$$\frac{2}{7},\frac{7}{2} \qquad \frac{3}{4},\frac{4}{3} \qquad \frac{6}{1},\frac{1}{6}$$

Reciprocals are used in division of fractions.

Division of Fractions
To divide a first fraction by a second fraction, multiply the first fraction by the reciprocal of the second fraction. Reduce if possible.

This method is sometimes called the “invert and multiply” method.

Sample Set B

Perform the following divisions.

Example:

$\frac{1}{3} \div \frac{3}{4}$. The divisor is $\frac{3}{4}$. Its reciprocal is $\frac{4}{3}$.

$$\begin{aligned} \frac{1}{3} \div \frac{3}{4} &= \frac{1}{3} \cdot \frac{4}{3} \\ &= \frac{1 \cdot 4}{3 \cdot 3} \\ &= \frac{4}{9} \end{aligned}$$

Example:

$\frac{3}{8} \div \frac{5}{4}$. The divisor is $\frac{5}{4}$. Its reciprocal is $\frac{4}{5}$.

$$\begin{aligned} \frac{3}{8} \div \frac{5}{4} &= \frac{3}{8} \cdot \frac{4}{5} \\ &= \frac{3}{2 \cdot 2 \cdot 2} \cdot \frac{2 \cdot 2}{5} \\ &= \frac{3}{\cancel{2} \cdot \cancel{2} \cdot 2} \cdot \frac{\cancel{2} \cdot \cancel{2}}{5} \quad 2 \text{ is a common factor.} \\ &= \frac{3 \cdot 1}{2 \cdot 5} \\ &= \frac{3}{10} \end{aligned}$$

Example:

$\frac{5}{6} \div \frac{5}{12}$. The divisor is $\frac{5}{12}$. Its reciprocal is $\frac{12}{5}$.

$$\begin{aligned} \frac{5}{6} \div \frac{5}{12} &= \frac{5}{6} \cdot \frac{12}{5} \\ &= \frac{5}{2 \cdot 3} \cdot \frac{2 \cdot 2 \cdot 3}{5} \\ &= \frac{\cancel{5}}{\cancel{2} \cdot \cancel{3}} \cdot \frac{\cancel{2} \cdot 2 \cdot \cancel{3}}{\cancel{5}} \\ &= \frac{1 \cdot 2}{1} \\ &= 2 \end{aligned}$$

Addition and Subtraction of Fractions

Fractions with Like Denominators
To add (or subtract) two or more fractions that have the same denominators, add (or subtract) the numerators and place the resulting sum over the common denominator. Reduce if possible.

CAUTION

Add or subtract only the numerators. Do **not** add or subtract the denominators!

Sample Set C

Find the following sums.

Example:

$\frac{3}{7} + \frac{2}{7}$. The denominators are the same. Add the numerators and place the sum over 7.

$\frac{3}{7} + \frac{2}{7} = \frac{3+2}{7} = \frac{5}{7}$

Example:

$\frac{7}{9} - \frac{4}{9}$. The denominators are the same. Subtract 4 from 7 and place the difference over 9.

$\frac{7}{9} - \frac{4}{9} = \frac{7-4}{9} = \frac{3}{9} = \frac{1}{3}$

Fractions can only be added or subtracted conveniently if they have like denominators.

Fractions with Unlike Denominators
To add or subtract fractions having unlike denominators, convert each fraction to an equivalent fraction having as the denominator the least common multiple of the original denominators.

The least common multiple of the original denominators is commonly referred to as the **least common denominator** (LCD). See Section ([link]) for the technique of finding the least common multiple of several numbers.

Sample Set D

Find each sum or difference.

Example:

$\frac{1}{6} + \frac{3}{4}$. The denominators are not alike. Find the LCD of 6 and 4.

$\begin{cases} 6 = 2 \cdot 3 \\ 4 = 2^2 \end{cases}$ The LCD is $2^2 \cdot 3 = 4 \cdot 3 = 12$.

Convert each of the original fractions to equivalent fractions having the common denominator 12.

$\frac{1}{6} = \frac{1 \cdot 2}{6 \cdot 2} = \frac{2}{12}$ $\quad \frac{3}{4} = \frac{3 \cdot 3}{4 \cdot 3} = \frac{9}{12}$

Now we can proceed with the addition.

$$\begin{aligned} \frac{1}{6} + \frac{3}{4} &= \frac{2}{12} + \frac{9}{12} \\ &= \frac{2+9}{12} \\ &= \frac{11}{12} \end{aligned}$$

Example:

$\frac{5}{9} - \frac{5}{12}$. The denominators are not alike. Find the LCD of 9 and 12.

$\begin{cases} 9 = 3^2 \\ 12 = 2^2 \cdot 3 \end{cases}$ The LCD is $2^2 \cdot 3^2 = 4 \cdot 9 = 36$.

Convert each of the original fractions to equivalent fractions having the common denominator 36.

$\frac{5}{9} = \frac{5 \cdot 4}{9 \cdot 4} = \frac{20}{36}$ $\frac{5}{12} = \frac{5 \cdot 3}{12 \cdot 3} = \frac{15}{36}$

Now we can proceed with the subtraction.

$$\begin{aligned} \frac{5}{9} - \frac{5}{12} &= \frac{20}{36} - \frac{15}{36} \\ &= \frac{20-15}{36} \\ &= \frac{5}{36} \end{aligned}$$

Exercises

For the following problems, perform each indicated operation.

Exercise:

Problem: $\frac{1}{3} \cdot \frac{4}{3}$

Solution:

$\frac{4}{9}$

Exercise:

Problem: $\frac{1}{3} \cdot \frac{2}{3}$

Exercise:

Problem: $\frac{2}{5} \cdot \frac{5}{6}$

Solution:

$\frac{1}{3}$

Exercise:

Problem: $\frac{5}{6} \cdot \frac{14}{15}$

Exercise:

Problem: $\frac{9}{16} \cdot \frac{20}{27}$

Solution:

$\frac{5}{12}$

Exercise:

Problem: $\frac{35}{36} \cdot \frac{48}{55}$

Exercise:

Problem: $\frac{21}{25} \cdot \frac{15}{14}$

Solution:

$\frac{9}{10}$

Exercise:

Problem: $\frac{76}{99} \cdot \frac{66}{38}$

Exercise:

Problem: $\frac{3}{7} \cdot \frac{14}{18} \cdot \frac{6}{2}$

Solution:

1

Exercise:

Problem: $\frac{14}{15} \cdot \frac{21}{28} \cdot \frac{45}{7}$

Exercise:

Problem: $\frac{5}{9} \div \frac{5}{6}$

Solution:

$\frac{2}{3}$

Exercise:

Problem: $\frac{9}{16} \div \frac{15}{8}$

Exercise:

Problem: $\frac{4}{9} \div \frac{6}{15}$

Solution:

$\frac{10}{9}$

Exercise:

Problem: $\frac{25}{49} \div \frac{4}{9}$

Exercise:

Problem: $\frac{15}{4} \div \frac{27}{8}$

Solution:

$\frac{10}{9}$

Exercise:

Problem: $\frac{24}{75} \div \frac{8}{15}$

Exercise:

Problem: $\frac{57}{8} \div \frac{7}{8}$

Solution:

$\frac{57}{7}$

Exercise:

Problem: $\frac{7}{10} \div \frac{10}{7}$

Exercise:

Problem: $\frac{3}{8} + \frac{2}{8}$

Solution:

$\frac{5}{8}$

Exercise:

Problem: $\frac{3}{11} + \frac{4}{11}$

Exercise:

Problem: $\frac{5}{12} + \frac{7}{12}$

Solution:

1

Exercise:

Problem: $\frac{11}{16} - \frac{2}{16}$

Exercise:

Problem: $\frac{15}{23} - \frac{2}{23}$

Solution:

$\frac{13}{23}$

Exercise:

Problem: $\frac{3}{11} + \frac{1}{11} + \frac{5}{11}$

Exercise:

Problem: $\frac{16}{20} + \frac{1}{20} + \frac{2}{20}$

Solution:

$\frac{19}{20}$

Exercise:

Problem: $\frac{3}{8} + \frac{2}{8} - \frac{1}{8}$

Exercise:

Problem: $\frac{11}{16} + \frac{9}{16} - \frac{5}{16}$

Solution:

$\frac{15}{16}$

Exercise:

Problem: $\frac{1}{2} + \frac{1}{6}$

Exercise:

Problem: $\frac{1}{8} + \frac{1}{2}$

Solution:

$\frac{5}{8}$

Exercise:

Problem: $\frac{3}{4} + \frac{1}{3}$

Exercise:

Problem: $\frac{5}{8} + \frac{2}{3}$

Solution:

$\frac{31}{24}$

Exercise:

Problem: $\frac{6}{7} - \frac{1}{4}$

Exercise:

Problem: $\frac{8}{15} - \frac{3}{10}$

Solution:

$\frac{5}{6}$

Exercise:

Problem: $\frac{1}{15} + \frac{5}{12}$

Exercise:

Problem: $\frac{25}{36} - \frac{7}{10}$

Solution:

$\frac{-1}{180}$

Exercise:

Problem: $\frac{9}{28} - \frac{4}{45}$

Exercise:

Problem: $\frac{8}{15} - \frac{3}{10}$

Solution:

$\frac{7}{30}$

Exercise:

Problem: $\frac{1}{16} + \frac{3}{4} - \frac{3}{8}$

Exercise:

Problem: $\frac{8}{3} - \frac{1}{4} + \frac{7}{36}$

Solution:

$\frac{47}{18}$

Exercise:

Problem: $\frac{3}{4} - \frac{3}{22} + \frac{5}{24}$

Adding and Subtracting Fractions with Like and Unlike Denominators, and LCD

This module discusses how to add and subtract fractions with like denominators and how to find the least common denominator to allow addition and subtraction of fractions with unlike denominators.

Adding Fractions with Like Denominators

To add two or more fractions that have the same denominators, add the numerators and place the resulting sum over the **common denominator**. Reduce, if necessary.

Example:
Find the following sums.
$\frac{3}{7} + \frac{2}{7}$
The denominators are the same.
Add the numerators and place the sum over the common denominator, 7.
$\frac{3}{7} + \frac{2}{7} = \frac{3+2}{7} = \frac{5}{7}$

When necessary, reduce the result.

Example:
$\frac{1}{8} + \frac{3}{8} = \frac{1+3}{8} = \frac{4}{8} = \frac{1}{2}$

Note: We do not add denominators.

Example:

To see what happens if we **mistakenly** add the denominators as well as the numerators, let's add
$\frac{1}{2}$ and $\frac{1}{2}$.
Adding the numerators and **mistakenly** adding the denominators produces:
$\frac{1}{2} + \frac{1}{2} = \frac{1+1}{2+2} = \frac{2}{4} = \frac{1}{2}$
This means that $\frac{1}{2} + \frac{1}{2}$ is the same as $\frac{1}{2}$, which is preposterous! **We do not add denominators**.

Adding Fractions with Like Denominators - Exercises

Find the following sums.

Exercise:

Problem: $\frac{3}{8} + \frac{3}{8}$

Solution:

$\frac{6}{8} = \frac{3}{4}$

Exercise:

Problem: $\frac{7}{11} + \frac{4}{11}$

Solution:

$\frac{11}{11} = 1$

Exercise:

Problem: $\frac{15}{20} + \frac{1}{20} + \frac{2}{20}$

Solution:

$\frac{18}{20} = \frac{9}{10}$

Subtracting Fractions with Like Denominators

To subtract two or more fractions that have the same denominators, subtract the numerators and place the resulting difference over the **common denominator**. Reduce, if necessary.

Example:
Find the following differences.
$\frac{3}{5} - \frac{1}{5}$
The denominators are the same.
Subtract the numerators and place the difference over the common denominator, 5.
$\frac{3}{5} - \frac{1}{5} = \frac{3-1}{5} = \frac{2}{5}$

When necessary, reduce the result.

Example:
$\frac{8}{6} - \frac{2}{6} = \frac{6}{6} = 1$

Note: We do not subtract denominators.

Example:
To see what happens if we **mistakenly** subtract the denominators as well as the numerators, let's subtract

$\frac{7}{15}$ - $\frac{4}{15}$.
Subtracting the numerators and **mistakenly** subtracting the denominators produces:
$\frac{7}{15}$ - $\frac{4}{15}$ = $\frac{7-4}{15-15}$ = $\frac{3}{0}$
We end up dividing by zero, which is undefined. **We do not subtract denominators.**

Subtracting Fractions with Like Denominators - Exercises

Find the following differences.

Exercise:

Problem: $\frac{5}{12}$ - $\frac{1}{12}$

Solution:

$\frac{4}{12}$ = $\frac{1}{3}$

Exercise:

Problem: $\frac{3}{16}$ - $\frac{3}{16}$

Solution:

Result is 0

Exercise:

Problem: $\frac{16}{5}$ - $\frac{1}{5}$ - $\frac{2}{5}$

Solution:

Result is $\frac{13}{5}$

Adding and Subtracting Fractions with Unlike Denominators

Basic Rule: Fractions can only be added or subtracted conveniently if they have like denominators.

To see why this rule makes sense, let's consider the problem of adding a quarter and a dime.

A quarter is $\frac{1}{4}$ of a dollar.

A dime is $\frac{1}{10}$ of a dollar.

We know that 1 quarter + 1 dime = 35 cents. How do we get to this answer by adding $\frac{1}{4}$ and $\frac{1}{10}$?

We convert them to quantities of the same denomination.

A quarter is equivalent to 25 cents, or $\frac{25}{100}$.

A dime is equivalent to 10 cents, or $\frac{10}{100}$.

By converting them to quantities of the same denomination, we can add them easily:

$\frac{25}{100} + \frac{10}{100} = \frac{35}{100}$.

Same denomination $\rightarrow$ same denominator

If the denominators are not the same, make them the same by building up the fractions so that they both have a common denominator. A common denominator can always be found by multiplying all the denominators, but it is not necessarily the Least Common Denominator.

Least Common Denominator (LCD)

The LCD is the smallest number that is evenly divisible by all the denominators.

It is the **least common multiple** of the denominators.

The LCD is the product of all the **prime factors** of all the denominators, each factor taken the greatest number of times that it appears in any single denominator.

Finding the LCD

Example:
Find the sum of these unlike fractions.
$\frac{1}{12} + \frac{4}{15}$
Factor the denominators:
$12 = 2 \times 2 \times 3$
$15 = 3 \times 5$
What is the greatest number of times the prime factor 2 appear in any single denominator? **Answer: 2times.** That is the number of times the prime factor 2 will appear as a factor in the LCD.
What is the greatest number of times the prime factor 3 appear in any single denominator? **Answer: 1 time.** That is the number of times the prime factor 3 will appear as a factor in the LCD.
What is the greatest number of times the prime factor 5 appear in any single denominator? **Answer: 1 time.** That is the number of times the prime factor 5 will appear as a factor in the LCD.
So we assemble the LCD by multiplying each prime factor by the number of times it appears in a single denominator, or:
$2 \times 2 \times 3 \times 5 = 60$
60 is the Least Common Denominator (the Least Common Multiple of 12 and 15).

Building up the Fractions

To create fractions with like denominators, we now multiply the numerators by whatever factors are missing when we compare the original denominator to the new LCD.

Example:
In the fraction $\frac{1}{12}$, we multiply the denominator 12 by 5 to get the LCD of 60. Therefore we multiply the numerator 1 by the same factor (5).
$\frac{1}{12} \times \frac{5}{5} = \frac{5}{60}$
Similarly,
$\frac{4}{15} \times \frac{4}{4} = \frac{16}{60}$

Adding the Built Up Fractions

Example:
We can now add the two fractions because they have like denominators:
$\frac{5}{60} + \frac{16}{60} = \frac{21}{60}$
Reduce the result: $\frac{21}{60} = \frac{7}{20}$

Adding and Subtracting Fractions with Unlike Denominators - Exercises

Find the following sums and differences.

Exercise:

Problem: $\frac{1}{6} + \frac{3}{4}$

Solution:

Result is $\frac{11}{12}$

Exercise:

Problem: $\frac{5}{9} - \frac{5}{12}$

Solution:

Result is $\frac{5}{36}$

Exercise:

Problem: $\frac{15}{16} + \frac{1}{2} - \frac{3}{4}$

Solution:

Result is $\frac{35}{16}$

Module Review Exercises

Exercise:

Problem: $\frac{9}{15} + \frac{4}{15}$

Solution:

Result is $\frac{13}{15}$

Exercise:

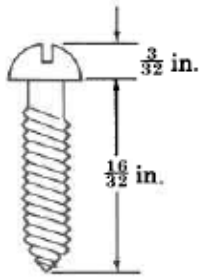
Problem: $\frac{7}{10}$ - $\frac{3}{10}$ + $\frac{11}{10}$

Solution:

Result is $\frac{15}{10}$ (reduce to 1 $\frac{1}{2}$)

Exercise:

Problem: Find the total length of the screw in this diagram:



Solution:

Total length is $\frac{19}{32}$ in.

Exercise:

Problem: $\frac{5}{2}$ + $\frac{16}{2}$ - $\frac{3}{2}$

Solution:

Result is $\frac{18}{2}$ (reduce to 9)

Exercise:

Problem: $\frac{3}{4}$ + $\frac{1}{3}$

Solution:

Result is $\frac{13}{12}$

Exercise:

Problem:

Two months ago, a woman paid off $\frac{3}{24}$ of a loan. One month ago, she paid off $\frac{4}{24}$ of the loan. This month she will pay off $\frac{5}{24}$ of the total loan. At the end of this month, how much of her total loan will she have paid off?

Solution:

She will have paid off $\frac{12}{24}$, or $\frac{1}{2}$ of the total loan.

Exercise:

Problem: $\frac{8}{3}$ - $\frac{1}{4}$ + $\frac{7}{36}$

Solution:

Result is $\frac{94}{36}$ (reduce to $\frac{47}{18}$)

Function Concepts -- Introduction
This module provides an introduction to the concept of functions.

The unit on functions is the most important in the Algebra II course, because it provides a crucial transition point. Roughly speaking…

- Before Algebra I, math is about **numbers**.
- Starting in Algebra I, and continuing into Algebra II, math is about **variables**.
- Beginning with Algebra II, and continuing into Calculus, math is about **functions**.

Each step builds on the previous step. Each step expands the ability of mathematics to model behavior and solve problems. And, perhaps most crucially, each step can be frightening to a student. It can be very intimidating for a beginning Algebra student to see an entire page of mathematics that is covered with letters, with almost no numbers to be found!

Unfortunately, many students end up with a very vague idea of what variables are ("That's when you use letters in math") and an even more vague understanding of functions ("Those things that look like or something"). If you leave yourself with this kind of vague understanding of the core concepts, the lessons will make less and less sense as you go on: you will be left with the feeling that "I just can't do this stuff" without realizing that the problem was all the way back in the idea of a variable or function.

The good news is, variables and functions both have very specific meanings that are not difficult to understand.

Function Concepts -- What is a Variable?
This module defines what a variable is and what it does.

A variable is a letter that stands for a number you don't know, or a number that can change.

A few examples:

Example:
Good Examples of Variable Definitions

- "Let p be the number of people in a classroom."
- "Let A be John's age, measured in years."
- "Let h be the number of hours that Susan has been working."

In each case, the letter stands for a very specific number. However, we use a letter instead of a number because we don't know the specific number. In the first example above, different classrooms will have different numbers of people (so p can be different numbers in different classes); in the second example, John's age is a specific and well-defined number, but we don't know what it is (at least not yet); and in the third example, h will actually change its value every hour. In all three cases, we have a good reason for using a letter: it represents a number, but we cannot use a specific number such as "–3" or " $4\frac{1}{2}$ ".

Example:
Bad Examples of Variable Definitions

- "Let n be the nickels."
- "Let M be the number of minutes in an hour."

The first error is by far the most common. Remember that a variable always stands for a number. “The nickels” are not a number. Better definitions would be: “Let n be the number of nickels” or “Let n be the total value of the nickels, measured in cents” or “Let n be the total mass of the nickels, measured in grams.”

The second example is better, because “number of minutes in an hour” is a number. But there is no reason to call it “The Mysterious Mr. M” because we already know what it is. Why use a letter when you just mean “60”?

Bad variable definitions are one of the most common reasons that students get stuck on word problems—or get the wrong answer. The first type of error illustrated above leads to variable confusion: n will end up being used for “number of nickels” in one equation and “total value of the nickels” in another, and you end up with the wrong answer. The second type of error is more harmless—it won’t lead to wrong answers—but it won’t help either. It usually indicates that the student is asking the wrong question (“What can I assign a variable to?”) instead of the right question (“What numbers do I need to know?”)

Variables aren’t all called x. Get used to it.

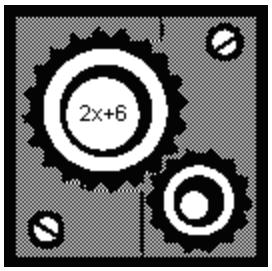
Many students expect all variables to be named x, with possibly an occasional guest appearance by y. In fact, variables can be named with practically any letter. Uppercase letters, lowercase letters, and even Greek letters are commonly used for variable names. Hence, a problem might start with “Let H be the home team’s score and V be the visiting team’s score.”

If you attempt to call both of these variables x, it just won’t work. You could in principle call one of them x and the other y, but that would make it more difficult to remember which variable goes with which team. It is important to become comfortable using a wide range of letters. (I do, however, recommend avoiding the letter o whenever possible, since it looks like the number 0.)

Function Concepts -- What is a Function?

A function is neither a number nor a variable: it is a **process for turning one number into another**. For instance, “Double and then add 6” is a function. If you put a 4 into that function, it comes out with a 14. If you put a $\frac{1}{2}$ into that function, it comes out with a 7.

The traditional image of a function is a machine, with a slot on one side where numbers go in and a slot on the other side where numbers come out.

$5 \rightarrow$		$\rightarrow 16$

A number goes in. A number comes out. The function is the machine, the process that turns 4 into 14 or 5 into 16 or 100 into 206.

The point of this image is that the function is not the numbers, but the machine itself—the **process**, not the results of the process.

The primary purpose of “The Function Game” that you play on Day 1 is to get across this idea of a numerical process. In this game, one student (the “leader”) is placed in the role of a function. “Whenever someone gives you a number, you double that number, add 6, and give back the result.” It should be very clear, as you perform this role, that you are **not** modeling a number, a variable, or even a list of numbers. You are instead modeling a process—or an algorithm, or a recipe—for turning numbers into other numbers. That is what a function is.

The function game also contains some more esoteric functions: “Respond with –3 no matter what number you are given,” or “Give back the lowest prime number that is greater than or equal to the number you were given.” Students playing the function game often ask “Can a function **do** that?” The answer is always yes (with one caveat mentioned below). So another purpose of the function game is to expand your idea of what a function can do. Any process that consistently turns numbers into other numbers, is a function.

By the way—having defined the word “function” I just want to say something about the word “equation.” An “equation” is when you “equate” two things—that is to say, set them equal. So $x^2 - 3$ is a function, but it is not an equation. $x^2 - 3 = 6$ is an equation. An “equation” always has an equal sign in it.

Function Concepts -- Four Ways to Represent a Function
This module discusses how it is possible to describe functions in four different ways: graphically, verbally, algebraically, and numerically.

Modern Calculus texts emphasize that a function can be expressed in four different ways.

1. **Verbal** - This is the first way functions are presented in the function game: “Double and add six.”
2. **Algebraic** - This is the most common, most concise, and most powerful representation: $2x + 6$. Note that in an algebraic representation, the input number is represented as a variable (in this case, an x).
3. **Numerical** - This can be done as a list of value pairs, as $(4\ 14)$ — meaning that if a 4 goes in, a 14 comes out. (You may recognize this as $(x\ y)$ points used in graphing.)
4. **Graphical** - This is discussed in detail in the section on graphing.

These are **not** four different types of functions: they are four different views of the same function. One of the most important skills in Algebra is converting a function between these different forms, and this theme will recur in different forms throughout the text.

Why are functions so important that they form the heart of math from Algebra II onward?

Functions are used whenever **one variable depends on another variable.** This relationship between two variables is the most important in mathematics. It is a way of saying “If you tell me what x is, I can tell you what y is.” We say that y “depends on” x, or y “is a function of” x.

A few examples:

Example:
Function Concepts -- Functions in the Real World

- "The area of a circle depends on its radius."
- "The amount of money Alice makes depends on the number of hours she works."
- “Max threw a ball. The height of the ball depends on how many seconds it has been in the air.”

In each case, there are two variables. Given enough information about the scenario, you could assert that **if you tell me this variable, I will tell you that one**. For instance, suppose you know that Alice makes $100 per day. Then we could make a chart like this.

If Alice works this many days...	**...she makes this many dollars**

If Alice works this many days...	...she makes this many dollars
0	0
1	100
1½	150
8	800

If you tell me how long she has worked, I will tell you how much money she has made. Her earnings “depend on” how long she works.

The two variables are referred to as the **dependent variable** and the **independent variable**. The dependent variable is said to “depend on” or “be a function of” the independent variable. “The height of the ball is a function of the time.”

Example:
Bad Examples of Functional Relationships

- "The number of Trojan soldiers depends on the number of Greek soldiers."
- "The time depends on the height of the ball."

The first of these two examples is by far the most common. It is simply not true. There may be a relationship between these two quantities—for instance, the **sum** of these two variables might be the total number of soldiers, and the **difference** between these two quantities might suggest whether the battle will be a fair one. But there is no **dependency** relationship—that is, no way to say “If you tell me the number of Greek

soldiers, I will tell you the number of Trojan soldiers”—so this is not a function.

The second example is subtler: it confuses the **dependent** and the **independent** variables. The height depends on the time, not the other way around. More on this in the discussion of “Inverse Functions".

Function Concepts -- Lines
This module discusses lines and their uses, and slope.

Most students entering Algebra II are already familiar with the basic mechanics of graphing lines. Recapping very briefly: the equation for a line is $y = \text{mx} + b$ where b is the y-intercept (the place where the line crosses the y-axis) and m is the slope. If a linear equation is given in another form (for instance, $4\text{x} + 2\text{y} = 5$), the easiest way to graph it is to rewrite it in $y = \text{mx} + b$ form (in this case, $y = -2\text{x} + 2\frac{1}{2}$).

There are two purposes of reintroducing this material in Algebra II. The first is to frame the discussion as **linear functions modeling behavior**. The second is to deepen your understanding of the important concept of slope.

Consider the following examples. Sam is a salesman—he earns a commission for each sale. Alice is a technical support representative—she earns $100 each day. The chart below shows their bank accounts over the week.

After this many days (t)	**Sam's bank account (S)**	**Alice's bank account (A)**
0 (*what they started with)	$75	$750
1	$275	$850
2	$375	$950
3	$450	$1,050
4	$480	$1,150

After this many days (t)	Sam's bank account (S)	Alice's bank account (A)
5	$530	$1,250

Sam has some extremely good days (such as the first day, when he made $200) and some extremely bad days (such as the second day, when he made nothing). Alice makes exactly $100 every day.

Let d be the number of days, S be the number of dollars Sam has made, and A be the number of dollars Alice has made. Both S and A are functions of time. But $s(t)$ is **not a linear function**, and $A(t)$**is a linear function**.

Linear Function

> A function is said to be "linear" if **every time the independent variable increases by 1, the dependent variable increases or decreases by the same amount**.

Once you know that Alice's bank account function is linear, there are only two things you need to know before you can predict her bank account on any given day.

- How much money she started with ($750 in this example). This is called the y**-intercept**.
- How much she makes each day ($100 in this example). This is called the **slope**.

y-intercept is relatively easy to understand. Verbally, it is where the function starts; graphically, it is where the line crosses the y-axis.

But what about slope? One of the best ways to understand the idea of slope is to convince yourself that all of the following definitions of slope are actually the same.

Definitions of Slope		
In our example	**In general**	**On a graph**
Each day, Alice's bank account increases by 100. So the slope is 100.	Each time the independent variable increases by 1, the dependent variable increases by the slope.	Each time you move to the **right** by 1, the graph goes **up** by the slope.
Between days 2 and 5, Alice earns $300 in 3 days. 300/3=100.Between days 1 and 3, she earns $200 in 2 days. 200/2=100.	Take any two points. The change in the dependent variable, **divided** by the change in the independent variable, is the slope.	Take any two points. The change in y divided by the change in x is the slope. This is often written as $\frac{\Delta y}{\Delta x}$, or as $\frac{\text{rise}}{\text{run}}$
The higher the slope, the faster Alice is making moey.	The higher the slope, the faster the dependent variable increases.	The higher the slope, the faster the graph rises as you move to the right.

So slope does not tell you where a graph is, but how quickly it is rising. Looking at a graph, you can get an approximate feeling for its slope without any numbers. Examples are given below.

A slope of 1: each time you go **over** 1,

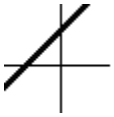
A steep slope of perhaps 3 or 4

A gentle slope of perhaps $\frac{1}{2}$.

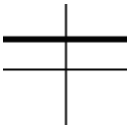
A horizontal line has a slope of 0: each time you go

This goes **down** as you move left to right. So the slope is negative. It is

you also go **up** 1



over 1, you don't go **up** at all!



steep: maybe a –2.

Function Concepts -- Composite Functions

You are working in the school cafeteria, making peanut butter sandwiches for today's lunch.

- The more classes the school has, the more children there are.
- The more children there are, the more sandwiches you have to make.
- The more sandwiches you have to make, the more pounds (lbs) of peanut butter you will use.
- The more peanut butter you use, the more money you need to budget for peanut butter.

...and so on. Each sentence in this little story is a function. Mathematically, if c is the number of classes and h is the number of children, then the first sentence asserts the existence of a function $h(c)$.

The principal walks up to you at the beginning of the year and says "We're considering expanding the school. If we expand to 70 classes, how much money do we need to budget? What if we expand to 75? How about 80?" For each of these numbers, you have to calculate each number from the previous one, until you find the final budget number.

# Classes		# Children		# Sandwiches		lb.		$$

But going through this process each time is tedious. What you want is one function that puts the entire chain together: "You tell me the number of classes, and I will tell you the budget."

# Classes		$$

This is a **composite function**—a function that represents in one function, the results of an entire **chain of dependent functions**. Since such chains are very common in real life,

finding composite functions is a very important skill.

How do you make a composite Function?

We can consider how to build composite functions into the function game that we played on the first day. Suppose Susan takes any number you give her, quadruples it, and adds 6. Al takes any number you give him and divides it by 2. Mathematically, we can represent the two functions like this:
Equation:

$$S(x) = 4\text{x} + 6$$

Equation:

$$A(x) = \frac{x}{2}$$

To create a chain like the one above, we give a number to Susan; she acts on it, and gives the resulting number to Al; and he then acts on it and hands back a third number.

$3 \to \text{Susan} \to S(3) = 18 \to \text{Al} \to A(18) = 9$

In this example, we are plugging $S(3)$—in other words, 18— into Al's function. In general, for any x that comes in, we are plugging $S(x)$ **into** $A(x)$. So we could represent the entire process as $A(S(x))$. This notation for composite functions is really nothing new: it means that you are plugging $S(x)$ into the A function.

But in this case, recall that $S(x) = 4\text{x} + 6$. So we can write:
Equation:

$$A(S(x)) = \frac{S(x)}{2} = \frac{4\text{x} + 6}{2} = 2\text{x} + 3$$

What happened? We've just discovered a shortcut for the entire process. When you perform the operation $A(S(x))$—that is, when you perform the Al function on the result of the Susan function—you are, in effect, doubling and adding 3. For instance, we saw earlier that when we started with a 3, we ended with a 9. Our composite function does this in one step:

$3 \to 2\text{x} + 3 \to 9$

Understanding the meaning of composite functions requires real thought. It requires understanding the idea that this variable depends on that variable, which in turn depends on the other variable; and how that idea is translated into mathematics. **Finding** composite functions, on the other hand, is a purely mechanical process—it requires practice, but no

creativity. Whenever you are asked for $f(g(x))$, just plug the $g(x)$ function into the $f(x)$ function and then simplify.

Example:
Building and Testing a Composite Function
$f(x) = x^2 - 4\text{x}$
$g(x) = x + 2$
What is $f(g(x))$?

- To find the composite, plug $g(x)$ into $f(x)$, just as you would with any number.

$f(g(x)) = (x + 2)^2 - 4(x + 2)$

- Then simplify.

$f(g(x)) = (x^2 + 4\text{x} + 4) - (4\text{x} + 8)$
$f(g(x)) = x^2 - 4$

- Let's test it. $f(g(x))$ means do g, then f. What happens if we start with $x = 9$?

$7 \to g(x) \to 7 + 2 = 9 \to f(x) \to (9)^2 - 4(9) = 45$

- So, if it worked, our **composite function** should do all of that in one step.

$7 \to x^2 - 4 = (7)^2 - 4 = 45$ It worked!

There is a different notation that is sometimes used for composite functions. This book will consistently use $f(g(x))$ which very naturally conveys the idea of "plugging $g(x)$ into $f(x)$." However, you will sometimes see the same thing written as $f \circ g(x)$, which more naturally conveys the idea of "doing one function, and then the other, in sequence." The two notations mean the same thing.

Function Concepts -- Inverse Functions
This module describes what inverse functions are and how they can be used.

Let's go back to Alice, who makes $100/day. We know how to answer questions such as "After 3 days, how much money has she made?" We use the function $m(t) = 100t$.

But suppose I want to ask the reverse question: “If Alice has made $300, how many hours has she worked?” This is the job of an inverse function. It gives the same relationship, but reverses the dependent and independent variables. $t(m) = m/100$. Given any amount of money, divide it by 100 to find how many days she has worked.

- **If a function answers the question:** “Alice worked this long, how much money has she made?” **then its inverse answers the question:** “Alice made this much money, how long did she work?"
- **If a function answers the question:** “I have this many spoons, how much do they weigh?” **then its inverse answers the question:** “My spoons weigh this much, how many do I have?”
- **If a function answers the question:** “How many hours of music fit on 12 CDs?” **then its inverse answers the question:** “How many CDs do you need for 3 hours of music?”

How do you recognize an inverse function?

Let’s look at the two functions above:
Equation:

$$m(t) = 100t$$

Equation:

$$t(m) = m/100$$

Mathematically, you can recognize these as inverse functions because they **reverse the inputs and the outputs**.

$3 \to m(t) = 100t \to 300$
$300 \to t(m) = m/100 \to 3$
✓ **Inverse functions**

Of course, this makes logical sense. The first line above says that “If Alice works 3 hours, she makes $300.” The second line says “If Alice made $300, she worked 3 hours.” It’s the same statement, made in two different ways.

But this “reversal” property gives us a way to test any two functions to see if they are inverses. For instance, consider the two functions:
Equation:

$$f(x) = 3\mathrm{x} + 7$$

Equation:

$$g(x) = \frac{1}{3}x - 7$$

They **look** like inverses, don’t they? But let’s test and find out.

$2 \to 3\mathrm{x} + 7 \to 13$
$13 \to \frac{3}{\mathrm{x}} - 7 \to \frac{13}{3} - 7 \to -\frac{8}{3}$
✗ **Not inverse functions**

The first function turns a 2 into a 13. But the second function does **not** turn 13 into 2. So these are not inverses.

On the other hand, consider:
Equation:

$$f(x) = 3\mathrm{x} + 7$$

Equation:

$$g(x) = \frac{1}{3}(x - 7)$$

Let’s run our test of inverses on these two functions.

$2 \to 3\mathrm{x} + 7 \to 13$
$13 \to \frac{1}{3}(x - 7) \to 2$
✓ **Inverse functions**

So we can see that these functions do, in fact, reverse each other: they are inverses.

A common example is the Celsius-to-Fahrenheit conversion:
Equation:

$$F(C) = \left(\frac{9}{5}\right)C + 32$$

Equation:

$$C(F) = \left(\frac{5}{9}\right)(F - 32)$$

where C is the Celsius temperature and F the Fahrenheit. If you plug $100^\circ C$ into the first equation, you find that it is $212^\circ F$. If you ask the second equation about $212^\circ F$, it of course converts that back into $100^\circ C$.

The notation and definition of an inverse function

The notation for the inverse function of $f(x)$ is $f^{-1}(x)$. This notation can cause considerable confusion, because it **looks like** an exponent, but it isn't. $f^{-1}(x)$ simply means "the inverse function of $f(x)$." It is defined formally by the fact that if you plug any number x into one function, and then plug the result into the other function, you get back where you started. (Take a moment to convince yourself that this is the same definition I gave above more informally.) We can represent this as a composition function by saying that $f(f^{-1}(x)) = x$.

Inverse Function

> $f^{-1}(x)$ is defined as the **inverse function** of $f(x)$ if it consistently reverses the $f(x)$ process. That is, if $f(x)$ turns a into b, then $f^{-1}(x)$ must turn b into a. More concisely and formally, $f^{-1}(x)$ is the inverse function of $f(x)$ if $f(f^{-1}(x)) = x$.

Finding an inverse function

In examples above, we saw that if $f(x) = 3x + 7$, then $f^{-1}(x) = \frac{1}{3}(x - 7)$. We also saw that the function $\frac{1}{3}x - 7$, which may have looked just as likely, did **not** work as an inverse function. So in general, given a function, how do you find its inverse function?

Remember that an inverse function reverses the inputs and outputs. When we graph functions, we always represent the incoming number as x and the outgoing number as y. So to find the inverse function, **switch the x and y values, and then solve for** y.

Example:
Building and Testing an Inverse Function

1. Find the inverse function of $f(x) = \frac{2x-3}{5}$
    - **a.**Write the function as $y = \frac{2x-3}{5}$
    - **b.**Switch the x and y variables. $x = \frac{2y-3}{5}$
    - **c.**Solve for y. $5x = 2y - 3$. $5x + 3 = 2y$. $\frac{5x+3}{2} = y$. So $f^{-1}(x) = \frac{5x+3}{2}$.
2. Test to make sure this solution fills the definition of an inverse function.
    - **a.**Pick a number, and plug it into the original function. $9 \to f(x) \to 3$.
    - **b.**See if the inverse function reverses this process. $3 \to f^{-1}(x) \to 9$. ✓ It worked!

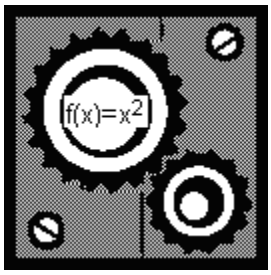
Were you surprised by the answer? At first glance, it seems that the numbers in the original function (the 2, 3, and 5) have been rearranged almost at random.

But with more thought, the solution becomes very intuitive. The original function $f(x)$ described the following process: **double a number, then subtract 3, then divide by 5**. To reverse this process, we need to reverse each step in order: **multiply by 5, then add 3, then divide by 2**. This is just what the inverse function does.

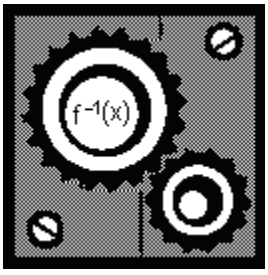
Some functions have no inverse function

Some functions have no inverse function. The reason is the rule of consistency.

For instance, consider the function $y = x^2$. This function takes both 3 and –3 and turns them into 9. No problem: a function is allowed to turn different **inputs** into the same **output**. However, what does that say about the inverse of this particular function? In order to fulfill the requirement of an inverse function, it would have to take 9, and turn it into both 3 and –3—which is the one and only thing that functions are **not** allowed to do. Hence, the inverse of this function would not be a function at all!

$3 \rightarrow$ $-3 \rightarrow$		$9 \rightarrow$ $9 \rightarrow$

If 3 goes in, 9 comes out. If –3 goes in, 9 also comes out. No problem:

$9 \rightarrow$ $9 \rightarrow$		$3 \rightarrow$ $-3 \rightarrow$

But its inverse would have to turn 9 into both 3 and –3. No function can do

this, so there is no inverse.

In general, any function that turns multiple inputs into the same output, does not have an inverse function.

What does that mean in the real world? If we can convert Fahrenheit to Celsius, we must be able to convert Celsius to Fahrenheit. If we can ask "How much money did Alice make in 3 days?" we must surely be able to ask "How long did it take Alice to make $500?" When would you have a function that cannot be inverted?

Let's go back to this example:

Recall the example that was used earlier: "Max threw a ball. The height of the ball depends on how many seconds it has been in the air." The two variables here are h (the height of the ball) and t (the number of seconds it has been in the air). The function $h(t)$ enables us to answer questions such as "After 3 seconds, where is the ball?"

The inverse question would be "At what time was the ball 10 feet in the air?" The problem with that question is, it may well have **two answers**!

The ball is here...	**...after this much time has elapsed**
10 ft	2 seconds (*on the way up)
10 ft	5 seconds (*on the way back down)

So what does that mean? Does it mean we can't ask that question? Of course not. We can ask that question, and we can expect to mathematically find the answer, or answers—and we will do so in the quadratic chapter. However, it does mean that **time is not a function of height** because such a

“function” would not be consistent: one question would produce multiple answers.